Journal of Combinatorial Theory, Series B **84**, 340–363 (2002)
doi:10.1006/jctb.2001.2088, available online at http://www.idealibrary.com on IDEAL®

# A Continuous Analogue of the Girth Problem

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Received December 22, 1999



without a cat! It's the most curious thing I ever saw in all my life!"
—Lewis Carroll, *Alice in Wonderland*

Let $A$ be the adjacency matrix of a $d$-regular graph of order $n$ and girth $g$ and $d=\lambda_1 \geqslant \cdots \geqslant \lambda_n$ its eigenvalues. Then $\sum_{j=2}^{n} \lambda_j^i = nt_i - d^i$, for $i=0, 1, \ldots, g-1$, where $t_i$ is the number of closed walks of length $i$ on the $d$-regular infinite tree. Here we consider distributions on the real line, whose $i$th moment is also $nt_i - d^i$ for all $i=0, 1, \ldots, g-1$. We investigate distributional analogues of several extremal graph problems involving the parameters $n$, $d$, $g$, and $\Lambda = \max |\lambda_2|, |\lambda_n|$. Surprisingly, perhaps, many similarities hold between the graphical and the distributional situations. Specifically, we show in the case of distributions that the least possible $n$, given $d$, $g$ is exactly the (trivial graph-theoretic) Moore bound. We also ask how small $\Lambda$ can be, given $d$, $g$, and $n$, and improve the best known bound for graphs whose girth exceeds their diameter. 

## 1. INTRODUCTION

Recall that the *girth* of a graph $G$ is the length of the shortest cycle in $G$. The construction of graphs with high girth (i.e., with no short cycles) is a notoriously difficult problem in extremal graph theory. The present work

[1] Supported in part by grants from the US–Israel Binational Science Fund and from the Israeli Academy of Science.



0095-8956/02 $35.00

has begun from an attempt to understand this problem from a new perspective. Previous work on this problem is based on two types of ideas: spectral analysis of the adjacency matrix of the graph in question and arithmetical arguments. The part that is based on spectral analysis usually does not take into account the fact that the relevant matrix represents a graph, so it is not too surprising that some of these questions can be addressed for general real symmetric matrices.

What is perhaps less expected, and indeed this is our point of departure, is that many of these considerations do not even need the matrix and can be carried out directly in terms of the eigenvalues. Moreover, the relevant equations involve sums of powers of the eigenvalues and can therefore be considered as conditions on the moments of distributions on the real line. This perspective allows us to invoke the theory of the Markov moment problem as a major tool in this area. The main focus of this paper is the analysis of these questions at the level of real distributions. We do not, however, lose sight of the graph-theoretic problem from which this investigation emerged. Our perspective allows us to tie these investigations to the spectral theory of graphs. Surprisingly, perhaps, we discover that many results previously known for graphs hold at the much greater generality of real distributions. We do encounter, however, cases where the phenomena are different. The quotation from [5] captures our response to the fact that so many problems and interesting phenomena remain even when the graphs are altogether absent.

We begin with a quick survey of what is known about the girth problem for graphs. Let $g = g(n, d)$ be the largest possible girth in a $d$-regular graph of order $\leqslant n$. If we consider $d \geqslant 3$ fixed and growing $n$, the best asymptotic estimates known are:

$$2\,\frac{\log n}{\log(d-1)}\cdot(1+o(1)) \geqslant g(n, d) \geqslant \frac{4}{3}\,\frac{\log n}{\log(d-1)}\cdot(1-o(1)).$$

The upper bound follows from a simple counting argument, and the lower bound was attained in [14] for infinitely many $d$'s. Despite the simplicity of the proof for the upper bound, no improvements are known on the asymptotic upper estimate for $g(n, d)$. To proceed with our discussion, we should reverse the above definition and let $n_G(d, g)$ be the least order $n$ of a $d$-regular graph with girth $g$. Then the aforementioned simple counting argument yields the so-called *Moore bound* (see [3, p. 180]),

$$n_G(d, g) \geqslant n_0(d, g), \tag{1}$$

where $n_0(d, g)$ is defined as

$$n_0(d, 2r+1) = 1 + d + d(d-1) + d(d-1)^2 + \cdots + d(d-1)^{r-1}$$
$$n_0(d, 2r) = 1 + d + d(d-1) + d(d-1)^2 + \cdots + d(d-1)^{r-2} + (d-1)^{r-1}$$

for odd and even values of $g$. A $d$-regular graph of girth $g$ with $n_0(d, g)$ vertices is called a *Moore graph* for odd $g$ and a *generalized polygon* if $g$ is even. Excluding trivial cases, we assume that $d \geqslant 3$ and $g \geqslant 5$. It is known (and requires considerable work) that Moore graphs exist only for $g = 5$ and this only for $d = 3$ or 7 and possibly 57. Generalized polygons exist only for $g \in \{4, 6, 8, 12\}$ (see [3, Theorem 23.6] and the survey [18]).

The known proof of this theorem uses algebraic properties of the adjacency matrix to show that the spectra of Moore graphs or generalized polygons are completely determined by the girth requirement. In fact, when one derives explicit formulas for the eigenvalues and their multiplicities, one finds that for almost all values of $g$ the multiplicities are not integral, and therefore no such graphs exist. Later works [1, 4, 10] that improve these bounds slightly (by 1 or 2) use similar methods.

Here we introduce an alternative approach to these problems. The condition that $G$ has girth $g$ determines the values of the traces $\mathrm{Tr}(A^k)$ for $k < g$, where $A$ is the adjacency matrix of $G$. Specifically, $A$ has to satisfy the system of equations

$$\mathrm{Tr}(A^k) = nt_k \qquad \text{for} \quad k = 0, 1, ..., g-1, \tag{2}$$

where $t_k$ is the number of closed walks of length $k$ in the $d$-regular infinite tree $\mathbf{T_d}$.

To formulate a *distributional* version of the problem, we rewrite (2) as

$$\sum_{i=2}^{n} \lambda_i^k = nt_k - d^k \qquad \text{for} \quad k = 0, 1, ..., g-1, \tag{3}$$

where $d = \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ are the eigenvalues of $A$. (The reason for moving $d$ to the right-hand side of the equation will soon be explained.) Observe that $\mathrm{Tr}(A^k)$ is the $k$th moment of $A$'s eigenvalues, so a distributional analogue of the above questions indeed suggests itself. If $\mu$ is a nonnegative measure on $[-d, d]$, the analogue to (3) is:

$$\int_{-d}^{d} x^k \, d\mu(x) = nt_k - d^k \qquad \text{for} \quad k = 0, 1, ..., g-1. \tag{4}$$

It is now clear why $d$ was demoted to the right-hand side of the equation: Otherwise, $n$ becomes a meaningless normalization factor. This choice is also motivated by considering the $d$-regular infinite tree $\mathbf{T_d}$—the ideal member in this class of graphs with infinite girth. Indeed, the spectral measure $\tau = \tau_d$ of $\mathbf{T_d}$ is supported on the interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$ and satisfies

$$\int_{-2\sqrt{d-1}}^{2\sqrt{d-1}} x^k \, d\tau(x) = t_k \qquad \text{for all} \quad k = 0, 1... .$$

We are thus led to the following: Given integers $g$ and $d$, define $n_D(d, g)$ as the least real $n$ for which there exists a nonnegative measure $\mu$ on the interval $[-d, d]$ that satisfies Eqs. (4).

In Section 3 we invoke the theory of the *Markov moment problem* [7, 11] to prove

THEOREM 1.1. *For every two natural numbers $d$, $g$:*

$$n_D(d, g) = n_0(d, g).$$

*Moreover, if $n = n_0(d, g)$, then there is a unique measure $\mu$ satisfying* (4).

Theorem 1.1 gives an alternative approach to the first part of the proof that no graph achieves equality in the Moore bound (1), for $g \notin \{3, 4, 5, 6, 8, 12\}$. Such a proof can completely avoid the use of the theory of distance regular graphs, since by Theorem 1.1 we know there is a unique measure $\mu$ satisfying (4). This measure is discrete and can be computed from the given moments. The second (arithmetic) part of the proof stays unchanged and consists of checking when $\mu$ is graphic, by requiring that the points in the support of $\mu$ have integral weights.

During the past two decades, much progress has been made in understanding the spectra of graphs, and in particular the relation between the second eigenvalue and the graph's expansion properties. It seems, therefore, natural to ask how this applies to problems related to moments of real distributions. Let us first ask how small the second eigenvalue $\Lambda$ can be, in a $d$-regular graph of order $n$. Call this number $\Lambda_G(d, n)$. The current best bound is due to Friedman [8]:

$$\Lambda_G(d, n) \geqslant 2\sqrt{d-1}\left(1 - O\left(\frac{1}{\log^2 n}\right)\right).$$

In Section 4 we define a distributional analogue $\Lambda_D(d, n)$ and compare it with $\Lambda_G(d, n)$. We show:

$$\Lambda_D(d, n) = 2\sqrt{d-1}\left(1 - \Theta\left(\frac{\log\log n}{\log n}\right)\right).$$

It turns out that for graphs of high girth, i.e., graphs whose girth is bigger than the diameter, Friedman's bound can be improved. The best known bound is due to Solé and Li [12, 17]:

$$\Lambda_G(d, g, n) \geqslant 2\sqrt{d-1}\left(1 - O\left(\frac{1}{g^2}\right)\right).$$

For distributions, we define $\Lambda_D(d, g, n)$ and give estimates for this quantity. It turns out that for graphs of high girth these estimates yield better lower bounds on $\Lambda_G(d, g, n)$, improving the lower bounds of Friedman [8] and of Solé and Li [12].

Perhaps the most significant part of spectral graph theory is the strong connection between the second eigenvalue and the graph's expansion properties. It has often been asked whether graphs of high girth necessarily have good expansion properties. This question must certainly be severely qualified, since for large $n$, high girth does not even imply connectivity: Take two disjoint copies of a graph with the largest possible girth (If a connected graph is sought, it is very easy to modify this example.) Thus the above statement can possibly hold only for $n \leqslant 2n_G(d, g)$. Because of the connection between the second eigenvalue and expansion, the relevant question is how large the second eigenvalue can be in a $d$-regular graph of order $n$ and girth $g$. The best known bound for this problem is due to Biggs [2]. In Section 5 we establish a bound on the distributional analogue of the same problem.

## 2. GENERAL BACKGROUND

### 2.1. *The Markov Moment Problem*

Let $\mu$ be a real nonnegative measure on the real line. As usual, the $k$th moment of $\mu$ is:

$$m_k = \int_{-\infty}^{\infty} x^k \, d\mu(x).$$

The question is which real sequences $m_0, m_1, \ldots$ can be attained this way. The following three theorems are part of the classical theory of the Markov moment problem and are immediate consequences of the Markov–Lukács theorem.

THEOREM 2.1 [11, Theorem 2.3, p. 62] or [6, Theorem 3.4, p. 15]). *There exists a nonnegative measure whose first moments are $m_0, \ldots, m_{2r}$ iff the matrix*

$$(m_{i+j})_{i,\, j=0,\, \ldots,\, r}$$

*is positive semidefinite.*

THEOREM 2.2 [11, Theorem 2.3, p. 62]. *There exists a nonnegative measure on the interval $[a, b]$, whose first $2r+1$ moments are $m_0, \ldots, m_{2r}$, iff both the matrix of Theorem 2.1 and the matrix*

$$((a+b)\, m_{i+j+1} - abm_{i+j} - m_{i+j+2})_{i,\, j=0,\, \ldots,\, r-1}$$

*are positive semidefinite.*

THEOREM 2.3 [11, Theorem 2.4, p. 63]. *There exists a nonnegative measure on the interval $[a, b]$, whose first $2r+2$ moments are $m_0, \ldots, m_{2r+1}$, if both matrices*

$$(m_{i+j+1} - am_{i+j})_{i,\, j=0,\, \ldots,\, r}; \qquad (bm_{i+j} - m_{i+j+1})_{i,\, j=0,\, \ldots,\, r}$$

*are positive semidefinite.*

*Remark* 2.1. In the three theorems above, the specified matrices are singular iff there is exactly one nonnegative measure with the specified moments. In this case we say that the given moment sequence is *singular*, or *singular w.r.t.* $[a, b]$.

We abbreviate positive semidefinite by PSD and positive definite by PD.

DEFINITION 2.1 (Upper–lower principal representations [11, p. 77]). Consider a moment sequence $m_0, \ldots, m_k$ and an interval $[a, b]$.

1. For $k = 2v - 1$, a measure on $[a, b]$ whose first moments are $m_0, \ldots, m_k$ is called:
   - lower principal representation if all its weight is concentrated at $v$ interior points of $[a, b]$,
   - upper principal representation if all its weight is concentrated at $v - 1$ interior points of $[a, b]$ and at both endpoints $a, b$.

2. For $k=2\nu$, a measure on $[a,b]$ whose first moments are $m_0, \ldots, m_k$ is called:

   - lower principal representation if all its weight is concentrated at $\nu$ interior points of $[a,b]$ and at the endpoint $a$,
   - upper principal representation if all its weight is concentrated at $\nu$ interior points of $[a,b]$ and at the endpoint $b$.

THEOREM 2.4 [11, Theorem 5.1, p. 86]. *For every moment sequence $m_0, \ldots, m_k$ which is nonsingular on $[a,b]$, there exists exactly one lower principal representation and exactly one upper principal representation.*

THEOREM 2.5 [11, Theorem 1.1, p. 109]. *For every moment sequence $m_0, \ldots, m_k$ which is nonsingular on $[a,b]$, there exists a measure on $[a,b]$ whose first moments are $m_0, \ldots, m_k$ that simultaneously maximizes (resp. minimizes) all moments $m_l$ for $l>k$. This measure is the upper (resp. lower) principal representation.*

### 2.2. *The d-Regular Infinite Tree*

Let $t_i$ be the number of closed walks of length $i$ in the infinite $d$-regular tree $\mathbf{T_d}$. The generating function of the sequence $\{t_i\}$ is known (see for example [13, pp. 55–56]:

$$T(z)=\sum_{i=0}^{\infty} t_i z^i = \frac{2(d-1)}{2(d-1)-d(1-\sqrt{1-4(d-1)\,z^2}\,)}. \tag{5}$$

An asymptotic expression for the numbers $t_k$ is given by [15]:

$$t_{2k}=(1+o(1))\cdot\frac{4^k\cdot d(d-1)^{k+1}}{\pi k^{3/2}(d-2)^2}. \tag{6}$$

Also, as noted in [16, 17] the spectral measure $\tau=\tau_{v,v}$ of $\mathbf{T_d}$ with respect to a vertex $v$ is supported on $[-2\sqrt{d-1}\,, 2\sqrt{d-1}]$ and satisfies:

$$\forall i \geqslant 0 \qquad t_i=\int_{-2\sqrt{d-1}}^{2\sqrt{d-1}} x^k\, d\tau(x). \tag{7}$$

The explicit expression for this measure is:

$$d\tau(x)=\frac{d}{2\pi}\,\frac{\sqrt{4(d-1)-x^2}}{d^2-x^2}\,dx. \tag{8}$$

### 2.3. *The Moore Spectrum*

The Moore spectrum, denoted by $\sigma = \sigma_{d,g}$, is a measure that plays a major role in this paper. This measure can be defined in more than one way:

- In the theory of distance regular graphs, one can compute the spectrum of a graph achieving equality in the Moore bound (1) (see [3, p. 183]).
- It is the unique measure satisfying Eqs. (4) with equality for $n = n_0(d, g)$. The existence and uniqueness of this measure are guaranteed by Theorem 1.1.
- The upper principal representation of the moment sequence

$$n_0(d, g)\, t_i \qquad \text{for} \quad i = 0, 1, \ldots, g-1$$

on the interval $[-d, d]$. The existence of the upper principal representation follows from Theorem 2.4 and the observation that there is more than one measure whose first $g$ moments are these: for example, the measure $n_0(d, g)\,\tau$ of the tree $\mathbf{T_d}$ and the measure defined by placing a weight 1 on each of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of (the adjacency matrix of) any $d$-regular graph with girth at least $g$ and $n$ vertices.

The spectrum $\sigma$ is known (see [3, Proposition 23.4, p. 183]). For odd girth $g = 2r+1$, it consists of $d$ (with weight 1) and the $r$ numbers of the form $\lambda = 2\sqrt{d-1}\cos\alpha$, where $0 < \alpha < \pi$ is a solution for the equation:

$$\sqrt{d-1}\,\sin(r+1)\,\alpha + \sin r\alpha = 0.$$

(The weights of these numbers are also known explicitly, but we do not need them here.) These $r$ numbers are exactly the roots of the polynomial $P_r(x)$ defined in Section 3.1. One more fact that we need about $\sigma$ (and is not difficult to obtain) is that the minimal $\Lambda$ for which $[\Lambda, \Lambda]$ contains the entire support of $\sigma$, except the point $d$, is

$$\Lambda_0(d, g) = 2\sqrt{d-1}\,(1 - 2\pi^2 g^{-2} + \Theta(g^{-4})) \tag{9}$$

and that indeed $-\Lambda_0(d, g)$ is in the support of $\sigma$. Also, we define $\sigma'$ to be the Moore spectrum with the point $d$ omitted.

## 3. PROOF OF THEOREM 1.1: $n_D(d, g) = n_0(d, g)$

Let $m_k = nt_k - d^k$. We show that if $n < n_0(d, g)$ then the moment sequence $m_0, \ldots, m_{g-1}$ is not feasible; i.e., there is no real distribution on $[-d, d]$ whose moment sequence is $m_0, \ldots, m_{g-1}$. Furthermore, if $n = n_0(d, g)$ we show that this moment sequence is singular, and therefore by Remark 2.1, there is exactly one distribution with the required moments, namely the distribution $\sigma'$ defined in Section 2.3.

### 3.1. *The Case of Odd Girth*

Let $g = 2r+1$, and let:

$$T_k = (t_{i+j})_{i,j=0,\dots,k}, \qquad \bar{T}_k = (t_{i+j+2})_{i,j=0,\dots,k}, \qquad D_k = (d^{i+j})_{i,j=0,\dots,k}. \quad (10)$$

Theorem 2.2 states that our moment sequence can be realized by a non-negative measure on $[-d, d]$ iff two matrices are PSD. It is not hard to calculate that these two matrices are:

$$U_{r,n} = (m_{i+j})_{i,j=0,\dots,r} = nT_r - D_r,$$
$$V_{r,n} = (d^2 m_{i+j} - m_{i+j+2})_{i,j=0,\dots,r-1} = n \cdot (d^2 T_{r-1} - \bar{T}_{r-1}).$$

When $n = n_G(d, g)$, a graph of order $n$ and girth $g$ exists, so these two matrices are PSD for this value of $n$. This settles the problem for $V_{r,n}$: It is PSD for every $n \geqslant 0$. Thus, to determine $n_D(d, g)$ only the matrix $U_{r,n}$ should be considered. It also follows that we actually prove more than $n_D(d, g) = n_0(d, g)$ for odd girth. Namely, the proof implies that for odd $g$ there exists a nonnegative measure whose moments are $m_0, m_1, \dots, m_{g-1}$ iff $n \geqslant n_0(d, g)$, *regardless of the distribution's support*.

We should now find the values of $n$ for which the matrix $nT_r - D_r$ is PSD. We first observe that $T_r$ is PD. This follows from the fact that the moment sequence $t_0, t_1, \dots, t_{g-1}$ may be realized in more than one way as seen in Section 2.3.

In order to prove that the matrix $nT_r - D_r$ is PSD, we need to check that each of its principal minors has a nonnegative determinant. That is $|nT_k - D_k| \geqslant 0$ for $k = 0, \dots, r$. Since all these minors have the same form, we concentrate on the last one. We employ a well-known formula for the determinant of the sum of two matrices. Let $X, Y$ be two $n \times n$ matrices. Then

$$\det(X+Y) = \sum_{\varepsilon \in \{0,1\}^n} \det(z_1^{(\varepsilon_1)} z_2^{(\varepsilon_2)} \cdots z_n^{(\varepsilon_n)}),$$

where $z_i^{(0)}$, $z_i^{(1)}$ are the $i$th column of $X, Y$ respectively. Since $D_r$ has rank one, in the expansion of $|nT_r - D_r|$ terms with two columns or more from $D_r$ clearly vanish. Therefore,

$$|nT_r - D_r| = |T_r| \cdot n^r \cdot (n - \gamma),$$

where $\gamma$ is a constant. We show below that $|n_0(d, g)\, T_r - D_r| = 0$ by exhibiting an explicit vector in its kernel. It follows that $|nT_r - D_r| = |T_r| \cdot n^r \cdot (n - n_0(d, g))$, and therefore since $|T_r| > 0$, that $nT_r - D_r$ is PSD iff $n \geqslant n_0(d, g)$.

Let $P_k(x)$ be the degree-$k$ polynomial such that if $A$ is the adjacency matrix of a $d$-regular graph with $n$ vertices and girth $g \geqslant 2k+1$, then:

$$P_k(A)_{u,v} = \begin{cases} 1 & dist(u,v) \leqslant k \\ 0 & \text{otherwise.} \end{cases}$$

These polynomials play a crucial role in [1] and [2]. They can be defined through the following recurrence:

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x+1, \\ P_k(x)\cdot x &= P_{k+1}(x)+(d-1)\cdot P_{k-1}(x) \qquad \text{for all} \quad k \geqslant 1. \end{aligned} \tag{11}$$

Set $P_k(x) = \sum_{i=0}^{k} p_{k,i} x^i$. Although unnecessary for the discussion below, we mention the explicit form of the $p_{k,i}$:

$$p_{k,i} = \begin{cases} \left( \begin{array}{c} \left\lfloor \dfrac{k+i}{2} \right\rfloor \\ \left\lfloor \dfrac{k-i}{2} \right\rfloor \end{array} \right) \cdot (1-d)^{\left\lfloor \frac{k-i}{2} \right\rfloor} & \text{for} \quad \mathrm{i} = 0, 1, \ldots, k \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

(This can be easily verified by showing that the polynomials defined by (12) satisfy the recurrence (11).)

Now, let $0 \leqslant l \leqslant r$, and again let $A$ be the adjacency matrix of any $d$-regular graph with $n$ vertices and girth at least $2r+1$. Then:

$$\begin{aligned} \sum_{i=0}^{r} p_{r,i} \cdot n \cdot t_{i+l} &= \mathrm{Tr}(P_r(A)\, A^l) = \sum_{v \in V} (P_r(A)\, A^l)_{v,v} = \sum_{v,u} P_r(A)_{v,u}\, A^l_{u,v} \\ &= \sum_{v,u} A^l_{u,v} = \sum_{v} d^l = n \cdot d^l. \end{aligned} \tag{13}$$

The penultimate equality follows, since the all ones vector $\bar{1}$ is an eigenvector of $A^l$. Also,

$$\sum_{i=0}^{r} p_{r,i} \cdot d^{i+l} = d^l \cdot P_r(d) = d^l \cdot \sum_{u} P_r(A)_{v,u} = d^l \cdot n_0(d, 2r+1),$$

where the penultimate equality holds for any vertex $v$, and the last one holds because $n_0(d, 2r+1)$ is the number of vertices in a ball of radius $r$ around $v$.

Let $p = p^{(r)} = (p_{r,0}, ..., p_{r,r})^t$. Combining the two identities we get that $(U_{r,n} \cdot p)_l = (n - n_0(d, g)) \cdot d^l$, which shows that the kernel of $n_0(d, g)\, T_r - D_r$ is nonempty.

We note that much of what was done in this section easily extends to graphs that are distance regular of order $r$. In this context, the polynomials $P_k$ play an important role in [9, Lemma 2.3, Section 4].

### 3.2. *The Case of Even Girth*

Let $g = 2r+2$. Again, by Theorem 2.3, the moment sequence $m_0, m_1, ..., m_{g-1}$ is realizable by a distribution on $[-d, d]$ iff the matrices

$$U_{r,n} = (m_{i+j+1} + dm_{i+j})_{i,j=0,...,r-1} = (n \cdot (t_{i+j+1} + dt_{i+j}) - 2d^{i+j+1})_{i,j=0,...,r-1},$$
$$V_{r,n} = (-m_{i+j+1} + dm_{i+j})_{i,j=0,...,r-1} = (n \cdot (-t_{i+j+1} + dt_{i+j}))_{i,j=0,...,r-1}$$

are PSD. Once again, $V_{r,n}$ is PSD for every $n \geqslant 0$. For the case of even girth this again yields a stronger result and applies to distributions that are supported on $[-d, \infty)$.

The matrix $U_{r,n}$ is PSD iff $\det(U_{k,n}) \geqslant 0$ for all $k = 0, ..., r$. The argument used at the case of odd girth implies that there is exactly one nonzero value of $n$ for which $\det(U_{k,n}) = 0$, and the determinant is positive (resp. negative) for $n$ above (resp. below) this critical value. We show now that this critical value is $n_0(d, 2r+2)$, by exhibiting a vector in the kernel of $U_{r, n_0(d, 2r+2)}$.

To this end, let $Q_k(x) = \sum_{i=0}^r q_{k,i} x^i$ be the degree $k$ polynomial such that if $A$ is the adjacency matrix of a $d$-regular graph with $n$ vertices and girth $g \geqslant 2k+1$, then:

$$Q_k(A)_{u,v} = \begin{cases} 1 & \mathit{dist}(u, v) \leqslant k \text{ and } \mathit{dist}(u, v) + k \text{ is even} \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Note that:

$$q_{k,i} = \begin{cases} p_{k,i} & i+k \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

let $0 \leqslant l \leqslant r+1$. Then

$$\sum_{i=0}^r q_{r,i} \cdot n \cdot t_{i+l} = \operatorname{Tr}(Q_r(A)\, A^l) = \sum_{v \in V} (Q_r(A)\, A^l)_{v,v} = \sum_{v,u} Q_r(A)_{v,u}\, A^l_{u,v}$$
$$= \begin{cases} \sum_{v,u} A^l_{u,v} & l+r \text{ is even} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} n \cdot d^l & l+r \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\sum_{i=0}^{r} q_{r,i} \cdot d^{i+l} = d^l \cdot \begin{cases} 1+d \sum_{j=0}^{(r-2)/2} (d-1)^{2j+1} & r \text{ is even} \\ d \sum_{j=0}^{(r-1)/2} (d-1)^{2j} & r \text{ is odd} \end{cases} = d^l \cdot \frac{n_0(2r+2)}{2},$$

where the last equality follows from a straightforward calculation.

Finally,

$$\sum_{i=0}^{r} q_{r,i} \cdot (U_{r,n})_{i,l} = \sum_{i=0}^{r} q_{r,i} \cdot (n \cdot (t_{i+l+1} + dt_{i+l}) - 2d^{i+l+1})$$
$$= n \cdot d^{l+1} - 2d^{l+1} \cdot \frac{n_0(d, 2r+2)}{2} = (n - n_0(d, 2r+2)) \cdot d^{l+1},$$

which is indeed zero for $n = n_0(d, 2r+2)$.

## 3. A LOWER BOUND ON $\Lambda$

Given a $d$-regular graph $G$ of girth $g$, diameter $D$, and with eigenvalues $d = \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, we denote $\Lambda_G = \Lambda = \max_{i=2,\ldots,n} |\lambda_i|$. The best known lower bounds on $\Lambda$ are

$$\Lambda \geqslant 2\sqrt{d-1}\left(1 - \frac{2\pi^2}{D^2} + O\left(\frac{1}{D^4}\right)\right) \tag{15}$$

(Friedman [8])

$$\Lambda \geqslant 2\sqrt{d-1}\left(1 - \frac{2\pi^2}{g^2} + O\left(\frac{1}{g^4}\right)\right) \tag{16}$$

(Solé and Li [12, 17]).

The latter bound is the stronger iff $g > D$, but since $g \leqslant 2D$, even then the improvement is only in the coefficient of the second order term. Since $D = \Omega(\log n)$, the best known lower bound on $\Lambda$ as a function of $n$ and $d$ is therefore:

$$\Lambda \geqslant 2\sqrt{d-1}\left(1 - O\left(\frac{1}{\log n^2}\right)\right). \tag{17}$$

Here, we consider a distributional analogue (called $\Lambda_D$) of this problem. In Section 4.1 we show that

$$\Lambda_D(d, n) = 2\sqrt{d-1}\left(1-\Theta\left(\frac{\log\log n}{\log n}\right)\right),$$

separating it from (17). In Section 4.2 we seek lower bounds on $\Lambda_D$ when also the girth is known and show that

$$\Lambda_D(d, g, n) \geqslant \Lambda_0(d, g),$$

where $\Lambda_0(d, g)$ is $\Lambda$ for the Moore distribution given by (9). (This turns out to be exactly the bound given by Solé and Li in [12].) We also show that the function $\Lambda_D(d, g, n)$ is monotone in $n$ and that $\Lambda_D(d, g, n) > \Lambda_0(d, g)$ for any $n > n_0(d, g)$. It is interesting that for small girth (or large $n$) the distributional analogue $\Lambda_D$ is separated below from $\Lambda_G$ and so does not yield any interesting new bounds in the graphical context. In contrast, for large girth $\Lambda_D$ provides an improvement of the known lower bound on $\Lambda_G$.

*Remark* 4.1. Friedman's bound [8] can be slightly improved. We give a brief sketch of his original proof and our improvement. Consider the $d$-regular tree of depth $\lfloor \frac{D}{2} \rfloor - 1$, and let $\lambda_1$ be the largest eigenvalue of its adjacency matrix. The corresponding eigenfunction is symmetric and depends only on the distance from the root, so let $f_i$ be its value at distance $i$ from the root. Given a graph $G$ with diameter $D$, let $x$, $y$ be two vertices with $dist(x, y) = D$. Define the function $\tilde{f}$ so that its value at vertex $z$ is:

$$\tilde{f}(z) = \begin{cases} c \cdot f_i & \text{if} \quad dist(z, x) = i < \left\lfloor \frac{D}{2} \right\rfloor \\ -f_i & \text{if} \quad dist(z, y) = i < \left\lfloor \frac{D}{2} \right\rfloor \\ 0 & \text{otherwise.} \end{cases}$$

The constant $c$ is chosen so as to make $\tilde{f}$ orthogonal to the constant function 1. A calculation of $\tilde{f}$'s Rayleigh quotient implies that $\Lambda \geqslant \lambda_1$.

We can improve upon this argument and get $\Lambda \geqslant \Lambda_0(d, 2\lfloor \frac{D}{2} \rfloor - 1)$. The improved lower bound is obtained by replacing $f_i$ with $f'_i = P_i(\Lambda_0(d, 2\lfloor \frac{D}{2} \rfloor - 1))$ where $P_i$ are the polynomials defined in (11). One can verify that $f'_i$ decrease with $i$ for $i < \lfloor \frac{D}{2} \rfloor$, so Friedman's argument can be appropriately modified. This yields a better constant in the third term of (15). For small values of $n$ this improvement may be substantial.

### 4.1. *Arbitrary Girth*

Given the integers $d$ and $n$, let $\Lambda_G(d, n)$ be the smallest possible value of $\Lambda$ for a $d$-regular graph of order $n$. The best known lower bound is (17). We would like to find a reasonable analog for distributions. Consider the following inequalities, which obviously hold for the spectrum $\mu(x)$ of any $d$-regular graph of order $n$, when we exclude the eigenvalue $d$:

$$\int_{-\Lambda}^{\Lambda} x^k \, d\mu(x) \geqslant nt_k - d^k \qquad \text{for all} \qquad k \geqslant 0. \tag{18}$$

Also note that $\mu([-\Lambda, \Lambda]) = n-1$. Therefore, given a natural $d$ and a real $n \geqslant 1$, we ask for the smallest value of $\Lambda = \Lambda_D(d, n)$, so that there exists a measure $\mu$ on $[-\Lambda, \Lambda]$ satisfying the inequalities (18), and $\mu([-\Lambda, \Lambda]) = n-1$.

We claim that:

THEOREM 4.1.

$$\Lambda_D(d, n) = 2\sqrt{d-1}\left(1 - \Theta\left(\frac{\log \log n}{\log n}\right)\right).$$

Note that Eq. (17) implies that $\Lambda_G$ and $\Lambda_D$ are separated.

*Proof.* Observe that the measure concentrated at $\Lambda$ has moments that exceed those of any other measure on $[-\Lambda, \Lambda]$. Therefore, we may assume that the measure attaining the minimum $\Lambda_D$ is a mass of $n-1$ concentrated at $\Lambda$, and therefore $\Lambda_D$ is the minimum value of $\Lambda$ satisfying:

$$(n-1)\,\Lambda^k \geqslant nt_k - d^k \qquad \text{for all} \qquad k \geqslant 0.$$

Phrasing it differently:

$$\Lambda_D(d, n) = \sup_{k \geqslant 1} \left(\frac{nt_{2k} - d^{2k}}{n-1}\right)^{\frac{1}{2k}}. \tag{19}$$

In fact, this sup is attained because $nt_{2k} - d^{2k} < 0$, for $k$ beyond $\Omega(\log n)$.

To obtain a lower bound on $\Lambda_D(d, n)$, any specific value of $k$ for which $\frac{n}{2}\, t_{2k} > d^{2k}$ yields a lower bound for (19):

$$\Lambda_D(d, n) \geqslant \left(\frac{\frac{1}{2}\, nt_{2k}}{n-1}\right)^{\frac{1}{2k}} \geqslant \left(\frac{1}{2}\, t_{2k}\right)^{\frac{1}{2k}}.$$

Using (6) we know that for some constant $c$:

$$\Lambda_D(d, n) \geqslant \left(\frac{1}{2} t_{2k}\right)^{\frac{1}{2k}} \geqslant 2\sqrt{d-1} \cdot \left(\frac{c}{k^{3/2}}\right)^{\frac{1}{2k}} \geqslant 2\sqrt{d-1} \cdot \left(1 - O\left(\frac{\log k}{k}\right)\right). \tag{20}$$

This bound is optimized by the largest possible $k$ for which $\frac{n}{2} t_{2k} \geqslant d^{2k}$, namely:

$$\log n - 1 \geqslant 2k \log d - \log t_{2k}. \tag{21}$$

By (6),

$$\log t_{2k} = 2k \cdot (\log(2\sqrt{d-1} - o(1))),$$

and therefore we can choose $k = \Theta(\log n)$ satisfying (21). Substituting it into (20) we get the desired lower bound.

To prove an upper bound on (19):

$$\begin{aligned} \Lambda_D(d, n) &= \max \left(\frac{n t_{2k} - d^{2k}}{n-1}\right)^{\frac{1}{2k}} \\ &\leqslant \max_{\{k:\, n t_{2k} \geqslant d^{2k}\}} \left(\frac{n t_{2k}}{n-1}\right)^{\frac{1}{2k}} \\ &\leqslant \left(1 + \frac{1}{n-1}\right) \max_{\{k:\, n t_{2k} \geqslant d^{2k}\}} (t_{2k})^{\frac{1}{2k}}. \end{aligned}$$

Using (6), we get that for some constant $c$:

$$\Lambda_D(d, n) \leqslant \left(1 + \frac{1}{n-1}\right) 2\sqrt{d-1} \max_{\{k:\, cn4^k (d-1)^k / k^{3/2} \geqslant d^{2k}\}} \left(\frac{c}{k^{3/2}}\right)^{\frac{1}{2k}}.$$

In the last upper bound, the function being maximized increases with $k$, and since the maximum possible $k$ is $\Theta(\log n)$:

$$\begin{aligned} \Lambda_D(d, n) &\leqslant \left(1 + \frac{1}{n-1}\right) 2\sqrt{d-1} \left(1 - \Omega\left(\frac{\log k}{k}\right)\right) \\ &= 2\sqrt{d-1} \left(1 - \Omega\left(\frac{\log \log n}{\log n}\right)\right). \quad \blacksquare \end{aligned}$$

4.2. *Large Girth*

Given the natural numbers $d$, $g$, and $n$, let $\Lambda_G(d, g, n)$ be the smallest possible value of $\Lambda$ for a $d$-regular graph of order $n$ and girth $g$.

Again, we consider this question for distributions and define $\Lambda_D(d, g, n)$ as the minimal $\Lambda$ for which there is a distribution $\mu$ on $[-\Lambda, \Lambda]$ for which:

$$\int_{-\Lambda}^{\Lambda} x^k \, d\mu(x) = nt_k - d^k \qquad \text{for all} \qquad k = 0, 1, \ldots, g-1, \tag{22}$$

$$\int_{-\Lambda}^{\Lambda} x^k \, d\mu(x) \geqslant nt_k - d^k \qquad \text{for all} \qquad k \geqslant g. \tag{23}$$

Also, let $\Lambda'_D(d, g, n)$ be the same, but omit condition (23).

Let us restrict our discussion to the case of odd girth, $g = 2r+1$. We prove:

THEOREM 4.2. *The function $\Lambda_D(d, g, n)$ is defined iff $n \geqslant n_0(d, g)$ and satisfies:*

(1) $\Lambda_D(d, g, n_0(d, g)) = \Lambda_0(d, g) = 2\sqrt{d-1}\,(1 - 2\pi^2 g^{-2} + \Theta(g^{-4}))$.

(2) *$\Lambda_D(d, g, n)$ is increasing with $n$ and with $g$, and $\Lambda_D(d, g, n) > \Lambda_0(d, g)$ for all $n > n_0(d, g)$.*

(3) *$\Lambda_D(d, g, n) \leqslant 2\sqrt{d-1}$, for all (relevant) $g$ and $n$.*

(4) *$\Lambda_D(d, g, n) \geqslant 2\sqrt{d-1}\,(1 - O(\frac{\log\log n}{\log n}))$.*

*Proof.* By Theorem 2.2 we know that (22) is equivalent to the requirement that the two matrices

$$U_{r,n} = (nt_{i+j} - d^{i+j})_{i,j=0,1,\ldots,r}, \tag{24}$$

$$W_{r,n,\Lambda} = (n \cdot (\Lambda^2 t_{i+j} - t_{i+j+2}) + (d^2 - \Lambda^2)\, d^{i+j})_{i,j=0,1,\ldots,r-1} \tag{25}$$

are PSD.

The status of $U_{r,n}$ has already been determined in Section 3.1, namely $U_{r,n}$ is PSD iff $n \geqslant n_0(d, g)$. Henceforth, we therefore assume that indeed $n \geqslant n_0(d, g)$.

For $n = n_0(d, g)$, $U_{r,n}$ is singular and therefore the whole spectrum is determined. This spectrum is $\sigma'$ defined in Section 2.3, and the minimal $\Lambda$ for which its entire support except the point $d$ is contained in $[-\Lambda, \Lambda]$ is $\Lambda_0(d, g)$.

Since $\sigma$ is the upper principal representation, by Theorem 2.5 it maximizes all the moments $m_k$ for $k \geqslant g$. Recall that the moments of $n\tau$, the spectral measure of $\mathbf{T_d}$, are $nt_k$ for all $k$; hence the moments of $\sigma$ are $\geqslant nt_k$ for all $k \geqslant g$. We conclude that the measure $\sigma$ without the point $d$ satisfies (22) and (23), and therefore:

$$\Lambda_D(d, g, n_0(d, g)) = \Lambda_0(d, g).$$

This proves 1. We would like to show now that for all $d$, $g$, and $n \geqslant n_0(d, g)$:

$$\Lambda_0(d, g) \leqslant \Lambda_D(d, g, n) \leqslant 2\sqrt{d-1}\,. \tag{26}$$

The upper bound follows immediately from the fact that $(n - n_0(d, g))\tau + \sigma'$ satisfies both (22) and (23) for every value of $g$. To prove the lower bound consider $\Lambda'_D(d, g, n)$. Since obviously $\Lambda'_D \leqslant \Lambda_D$ it is enough to prove the lower bound for $\Lambda'_D$, which we turn to do. Since we are assuming $n \geqslant n_0(d, g)$, we need only consider the matrix $W_{r, n, \Lambda}$ (defined by (25)). First note that this matrix remains PSD when $n$ decreases. (This follows since the matrix $(d^2 - \Lambda^2) \cdot (d^{i+j})_{i, j = 0, 1, ..., r-1}$ is PSD, and the collection of $r \times r$ PSD matrices forms a cone.) It follows that if $\lambda = \Lambda'_D(d, g, n)$, then there is a measure satisfying (22) with parameters $d$, $g$, $n$, and $\lambda$, and therefore also a measure with parameters $d$, $g$, $\lambda$, and $n_0(d, g)$. But in the case $n = n_0(d, g)$, as we have seen, $\lambda$ cannot possibly be smaller than $\Lambda_0(d, g)$, which yields the desired lower bound.

We would like to prove also that $\Lambda_D(d, g, n)$ increases with $n$. (The monotonicity of $\Lambda'_D$ follows from an argument similar to the last argument, but the monotonicity of $\Lambda_D$ is more subtle.) For the following argument let us fix $d$, $g$, and $\Lambda \geqslant \Lambda_0(d, g)$. Let $\mathscr{T}$ be the set of all infinite sequences $(m_0, m_1, ...)$ that satisfy for the moment sequence $(m'_0, m'_1, ...)$ of some measure $\mu$ on $[-\Lambda, \Lambda]$, the equalities $m'_k = m_k$ for all $0 \leqslant k \leqslant g-1$, and $m'_k \geqslant m_k$ for $k \geqslant g$. It is easy to see that $\mathscr{T}$ is convex. We know that $\{n_0(d, g)\, t_k - d^k\}_{k=0}^{\infty} \in \mathscr{T}$. Therefore, if for some $n \geqslant n_0(d, g)$, the sequence $\{nt_k - d^k\}_{k=0}^{\infty}$ is also in $\mathscr{T}$, then by convexity, $\{n' t_k - d^k\}_{k=0}^{\infty} \in \mathscr{T}$ for every $n_0(d, g) \leqslant n' \leqslant n$. We deduce from this argument that indeed $\Lambda_D(d, g, n)$ is increasing with $n$. To prove that $\Lambda_D(d, g, n) > \Lambda_0(d, g)$ for all $n > n_0(d, g)$, we use Lemma 4.1 which gives a strictly increasing lower bound for $\Lambda'_D(d, g, n)$.

Furthermore, it is easy to see that $\Lambda_D(d, g, n)$ is increasing with $g$ and that $\Lambda_D(d, g, n) \geqslant \Lambda_D(d, n)$ where $\Lambda_D(d, n)$ is defined in Section 4.1. By Theorem 4.1 we deduce that $\Lambda_D(d, g, n) \geqslant 2\sqrt{d-1}\,(1 - O\,(\frac{\log \log n}{\log n}))$. ∎

LEMMA 4.1.

$$\Lambda'_D(d,g,n)^2 \geqslant \Lambda_0(d,g)^2 + \frac{(n-n_0(d,g))\cdot(d^2-\Lambda_0(d,g)^2)}{n\cdot f(d,g)-n_0(d,g)},$$

*where $f(d,g)$ is some function that is always bigger than* 1.

*Proof.* Let $P_r(x)$ be the polynomials defined in Section 3.1. As noted before, $-\Lambda_0(d,g)$ is a root of $P_r(x)$ and therefore we can define polynomials $S_r(x)$ by:

$$S_r(x) = \frac{P_r(x)}{x+\Lambda_0(d,g)} = \sum_{i=0}^{r-1} s_{r,i} x^i.$$

As we have seen, $\Lambda'_D(d,g,n)$ is the minimal $\Lambda$ for which the matrix $W = W_{r,n,\Lambda}$ (defined by (25)) is PSD. If $W$ is PSD, then specifically,

$$s^t \cdot W \cdot s \geqslant 0, \tag{27}$$

where $s = (s_{r,0}, s_{r,1}, ..., s_{r,r-1})^t$. To expand (27), we write $W = W_1 + W_2 + W_3$ where:

$$\begin{aligned} W_1 &= (d^2-\Lambda^2)(d^{i+j})_{i,j=0,...,r-1} \\ W_2 &= n\cdot(\Lambda_0^2 t_{i+j} - t_{i+j+2})_{i,j=0,...,r-1} \\ W_3 &= n\cdot(\Lambda^2-\Lambda_0^2)\cdot(t_{i+j})_{i,j=0,...,r-1}. \end{aligned}$$

We handle each of the three summands separately.

The first summand is:

$$s^t\cdot W_1\cdot s = (d^2-\Lambda^2)\, S_r(d)^2 = (d^2-\Lambda^2)\,\frac{P_r(d)^2}{(d+\Lambda_0)^2} = \frac{d^2-\Lambda^2}{(d+\Lambda_0)^2}\cdot n_0^2.$$

To evaluate the second summand, let $\tau$ be as before, the spectral measure of $\mathbf{T_d}$. Then:

$$\begin{aligned} s^t\cdot W_2\cdot s &= n\cdot \sum_{i,j=0}^{r-1} s_{r,i}s_{r,j}(\Lambda_0^2 t_{i+j} - t_{i+j+2}) \\ &= n\cdot \int (\Lambda_0^2 - x^2)\cdot S_r(x)^2\, d\tau(x) \\ &= n\cdot \int P_r(x)\, S_r(x)(\Lambda_0 - x)\, d\tau(x) \\ &= n\cdot \sum_{i=0}^{r-1} s_{r,i} \sum_{j=0}^{r} p_{r,j}(\Lambda_0 t_{i+j} - t_{i+j+1}). \end{aligned}$$

Using equality (13), we deduce:

$$s^t \cdot W_2 \cdot s = n \cdot S_r(d)(\Lambda_0 - d) = -n \cdot n_0 \cdot \frac{d-\Lambda_0}{d+\Lambda_0}.$$

To handle the third summand, first define

$$f = f(d, g) = \frac{(d+\Lambda_0)^2}{n_0} \cdot \sum_{i,j=0}^{r-1} s_{r,i} s_{r,j} t_{i+j} = \frac{(d+\Lambda_0)^2}{n_0} \int S_r(x)^2 \, d\mu(x), \quad (28)$$

where $\mu$ is any measure whose $k$th moment is $t_k$ for every $k \leqslant 2r-2$. using the definition of $f$ we can express the third summand as:

$$s^t \cdot W_3 \cdot s = n \cdot (\Lambda^2 - \Lambda_0^2) \cdot f \cdot n_0 / (d+\Lambda_0)^2.$$

Summing up, the requirement (27) yields:

$$0 \leqslant \frac{d^2 - \Lambda^2}{(d+\Lambda_0)^2} \cdot n_0^2 - n \cdot n_0 \cdot \frac{d-\Lambda_0}{d+\Lambda_0} + n \cdot (\Lambda^2 - \Lambda_0^2) \cdot f \cdot n_0 / (d+\Lambda_0)^2$$

$$0 \leqslant (d^2 - \Lambda^2) \cdot n_0 - n \cdot (d^2 - \Lambda_0^2) + n \cdot (\Lambda^2 - \Lambda_0^2) \cdot f.$$

Isolating $\Lambda$ produces the desired lower bound:

$$\Lambda_0^2 \cdot (nf - n) + d^2 \cdot (n - n_0) \leqslant \Lambda^2 \cdot (nf - n_0)$$

$$\Lambda_0^2 + \frac{(n-n_0) \cdot (d^2 - \Lambda_0^2)}{nf - n_0} \leqslant \Lambda^2.$$

Finally, we need to show that $f \geqslant 1$ so that the lower bound on $\Lambda$ will increase with $n$. We also give an upper bound on $f(d, g)$:

For the lower bound, use (28) with $\mu = \sigma_{d, 2r-1}$, the Moore spectrum. Since the weight of $d$ is $1/n_0(d, 2r-1)$, we get:

$$f(d, 2r+1) > \frac{(d+\Lambda_0)^2}{n_0(d, 2r+1)} \cdot \frac{S_r(d)^2}{n_0(d, 2r-1)} = \frac{n_0(d, 2r+1)}{n_0(d, 2r-1)} > d-1.$$

For the upper bound, recall that all the $r-1$ roots of $S_r(x)$ are in the interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$, and therefore $|S_r(x)| \leqslant (4\sqrt{d-1})^{r-1}$ for any $x$ in this interval. Applying (28) with the tree measure $\tau$ and using the fact that it is supported on $[-2\sqrt{d-1}, 2\sqrt{d-1}]$ we get:

$$\begin{aligned} f(d, 2r+1) &\leqslant \frac{(d+\Lambda_0)^2}{n_0(d, 2r+1)} \cdot [(4\sqrt{d-1})^{r-1}]^2 \\ &\leqslant \frac{(d+\Lambda_0)^2}{(d-1)^r} \cdot 16^{r-1} \cdot (d-1)^{r-1} \\ &\leqslant \frac{(d+\Lambda_0)^2}{d-1} \cdot 16^{r-1}. \quad \blacksquare \end{aligned}$$

As an illustration, consider the case $g=5$. Figure 1 shows the known lower bounds on $\Lambda_G(d, g, n)$, for this case, namely:

- $\Lambda_0(d, g) = \sqrt{d-\frac{3}{4}} + \frac{1}{2}$, the lower bound of Solé and Li (12).
- Friedman's lower bound (8).
- $\Lambda'_D(d, g, n)$. This is a lower bound on both $\Lambda_D(d, g, n)$ and $\Lambda_G(d, g, n)$, but is simpler from the computational point of view.

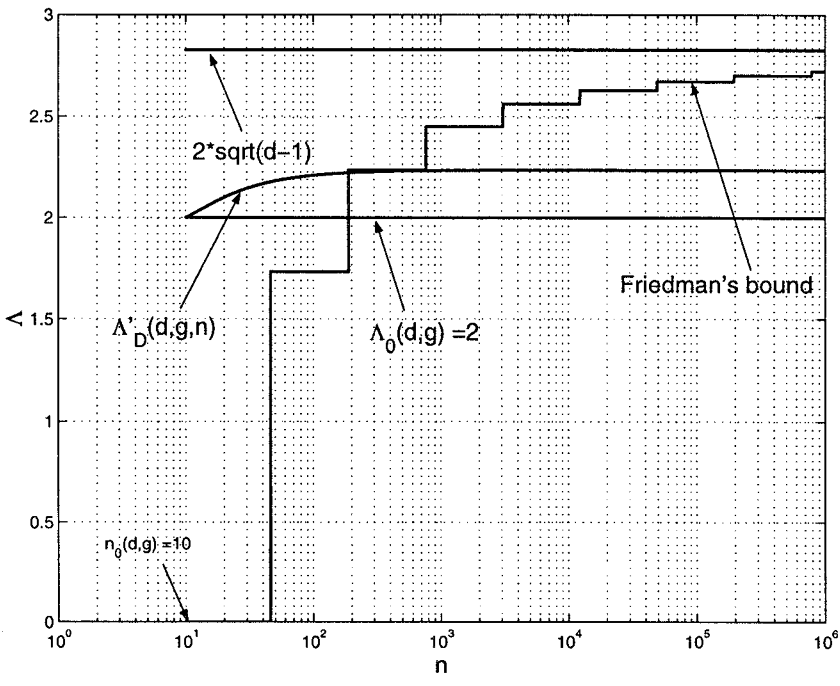


**FIG. 1.** $g=5, d=3$.

As seen in the figure, for a certain range of $n$'s the new bound improves the previously best known lower bound on $\Lambda$.

*Remark* 4.2. The definition of $\Lambda'_D(d, g, n)$ is convenient from the computational point of view, since its determination consists of finding the least $\Lambda$ for which (25) is PSD.

## 5. AN UPPER BOUND ON $\lambda_2$

In this section we study the following question: How large can the second largest eigenvalue be in a $d$-regular graph of girth $g$ and $n$ vertices? Or more generally, what is the minimal number of vertices $n_G(d, g, \lambda)$ in a $d$-regular graph having girth $g$ that has $\lambda$ as (some) eigenvalue.

As before, we extend the question to distributions; namely, we seek a measure $\mu$ on $[-d, d]$ that satisfies:

$$\int_{-d}^{d} x^k \, d\mu(x) = m_k = nt_k - d^k - \lambda^k \qquad \text{for} \quad k = 0, 1, \ldots, g-1. \tag{29}$$

We ask the question: Given $g$, $d$, and $\lambda$ what is the least real $n = n_D(d, g, \lambda)$ for which there exists a nonnegative measure $\mu$ on the interval $[-d, d]$ that satisfies Eqs. (29)?

It is interesting that there seems to be a gap between $n_G(d, g, \lambda)$ and $n_D(d, g, \lambda)$. Assume the girth is odd, $g = 2r+1$. The following lemmas recall a lower bound on $n_G(d, g, \lambda)$ due to Biggs and show a lower bound on $n_D(d, g, \lambda)$. In the following lemmas, $P_r$, $Q_r$ are the polynomials defined by (11), (14).

LEMMA 5.1 (Biggs [2]).

$$n_G(d, g, \lambda) - n_0(d, g) \geqslant |P_r(\lambda)|$$

LEMMA 5.2.

$$n_D(d, g, \lambda) - n_0(d, g) \geqslant \frac{P_r(\lambda)^2}{n_0(d, g)} \tag{30}$$

$$n_D(d, g, \lambda) - n_0(d, g+1)/2 \geqslant \frac{Q_r(\lambda)^2}{n_0(d, g+1)/2} \tag{31}$$

Note that the inequalities $|P_r(\lambda)| \leqslant n_0(d, g)$ and $|Q_r(\lambda)| \leqslant n_0(d, g+1)/2$ hold for all values $\lambda \in [-d, d]$, with equality for $\lambda = d$. It can be verified that the bound given by Lemma 5.2 is smaller than Biggs' lower bound.

We conjecture that the lower bound of Lemma 5.2 is nearly tight. This is based mainly on computer calculations for small values of $g$ and $d$. As an example, we show in Fig. 2 a graph of the lower bounds of Lemmas 5.1 and 5.2 and the exact value of $n_D(d, g, \lambda)$. If this conjecture turns out to be true then, as mentioned above, there is a significant separation between $n_D$ and $n_G$.

To determine the value of $n_D(d, g, \lambda)$, by Theorem 2.2 we have to determine the least $n$ for which the following two matrices are PSD:

$$U_{r,n,\lambda} = (m_{i+j})_{i,j=0,\dots,r} = (nt_{i+j} - d^{i+j} - \lambda^{i+j})_{i,j=0,\dots,r}$$
$$V_{r,n,\lambda} = (d^2 m_{i+j} - m_{i+j+2})_{i,j=0,\dots,r}$$
$$= (n \cdot (d^2 t_{i+j} - t_{i+j+2}) - (d^2 - \lambda^2) \cdot \lambda^{i+j})_{i,j=0,\dots,r}.$$

For the purpose of proving the lower bound, the latter condition will be ignored. Therefore, $n_D(d, g, \lambda)$ is at least the minimal $n$ for which $U_{r,n,\lambda}$ is PSD. Unfortunately, we do not know the exact solution for this problem for large $g$.

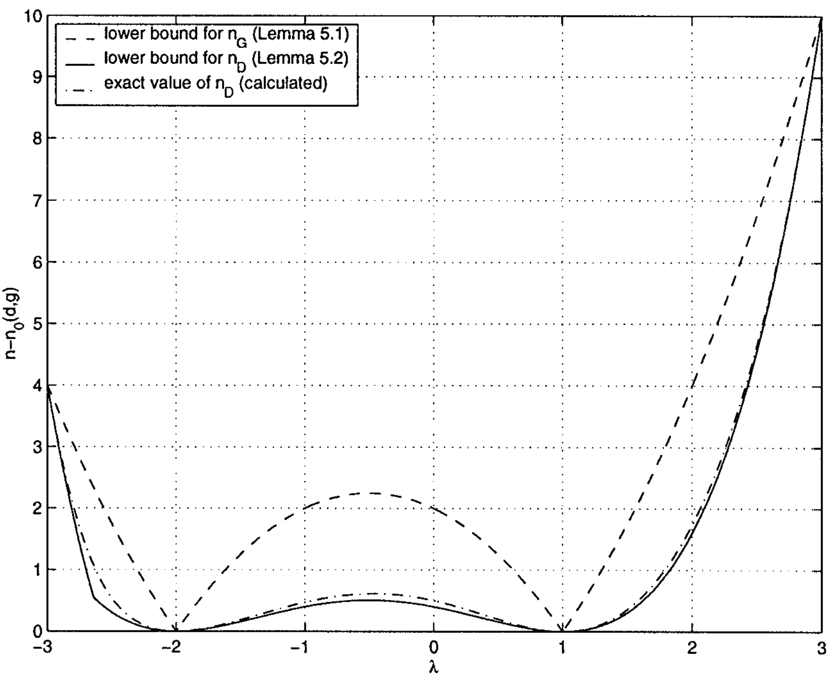


**FIG. 2.** $g = 5, d = 3$.

Here are two alternative proofs for Lemma 5.2.

*Proof* (of Lemma 5.2). Define $T_k$, $D_k$ as in (10), let $\Lambda_k = (\lambda^{i+j})_{i,j=0,\ldots,k}$, and let $p = p^{(r)}$ be as in Section 3.1. A necessary condition for the matrix $U_{r,n,\lambda} = nT_r - D_r - \Lambda_r$ to be PSD is that

$$p^t \cdot (nT_r - D_r - \Lambda_r) \cdot p \geqslant 0. \tag{32}$$

By Section 3.1 we know that:

$$p^t \cdot T_r \cdot p = P_r(d) = n_0(d, g)$$
$$p^t \cdot D_r \cdot p = P_r(d)^2 = n_0(d, g)^2.$$

Therefore, (32) yields

$$(n - n_0(d, g)) \cdot n_0(d, g) - P_r(\lambda)^2 \geqslant 0,$$

proving (30).

To prove (31), let $q = (q_{r,0}, \ldots, q_{r,r})^t$. Using the two identities from Section 3.2,

$$q^t \cdot T_r \cdot q = Q_r(d) = n_0(d, g+1)/2$$
$$q^t \cdot D_r \cdot q = Q_r(d)^2 = (n_0(d, g+1)/2)^2,$$

we get the bound (31) by a similar argument. ∎

The second proof of Lemma 5.2 is an immediate corollary of the following lemma:

LEMMA 5.3. *Let the measure $\mu$ satisfy for $g = 2r+1$:*

$$\int x^k \, d\mu(x) = nt_k - d^k \qquad \textit{for} \quad k = 0, 1, \ldots, g-1.$$

*Then:*

$$\int P_r(x)^2 \, d\mu(x) = (n - n_0(d, g)) \cdot n_0(d, g)$$

$$\int Q_r(x)^2 \, d\mu(x) = (n - n_0(d, g+1)/2) \cdot n_0(d, g+1)/2.$$

*Proof.* Using the results of Section 3.1,

$$\int P_r(x)^2 \, d\mu(x) = \sum_{i,j=0}^{r} p_{r,i} p_{r,j} (nt_{i+j} - d^{i+j})$$
$$= nP_r(d) - P_r(d)^2$$
$$= (n - n_0(d,g)) \cdot n_0(d,g).$$

In a very similar manner the second equality follows. ∎

Since $\mu(\{\lambda\}) \geqslant 1$, Lemma 5.2 follows.

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